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Linear subspaces of symmetric tensors whose non-zero elements have at least a prescribed rank

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Abstract We introduce large vector spaces M of multivariate homogeneous polynomials with a prescribed lower bound for the rank of each non-zero element of M.

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الملخص

نقدم فضاءات خطية واسعة M مكونة من كثيرات الحدود المتجانسة ومتعددة المتغيرات مع حد أدنى محدد مسبقاً لرتبة كل عنصر لاصفري من M.

1 Introduction

This paper has two stimuli. E. M. Gabidulin introduced the rank metric (instead of the Hamming metric) to define the minimum distance of a linear code [17,18]. Hence, it is nice to have large linear spaces of matrices or of tensors or of symmetric tensors such that each of its non-zero elements has at least a given rank, δ. In this paper, we consider linear spaces, W, of symmetric tensors, but we do not claim that our examples may be used to give nice codes, because in our examples all symmetric tensors $T \in W$ have rank $\geq \delta$ even over the algebraic closure of the base field. Hence, we do not use the Galois structure of finite fields, which should be essential to construct good Rank-Metric codes. The second input came from our previous work [4,5], in which certain vector spaces of homogeneous polynomials are a key tool (the projective spaces $W(O_1, \dots, O_k; d)$ defined below are the projectivations of the vector spaces we consider in this paper). To introduce our vector spaces of homogeneous polynomials, we recall the following classical set-up.

For all integers $m \geq 1$ and $d \geq 1$ let $\nu_d : \mathbb{P}^m \to \mathbb{P}^n$, $n = \binom{m+d}{m} - 1$, denote the order d Veronese embedding of $\mathbb{P}^m$, i.e., the embedding of $\mathbb{P}^m$ induced by the vector space of all homogeneous polynomials of degree d in $m+1$ variables. For each $P \in \mathbb{P}^n$ the rank or the symmetric tensor rank $r_{X_{m,d}}(P)$ of P is the minimal cardinality of a set $S \subset X_{m,d}$ such that $P \in \langle S \rangle$, where $\langle\ \rangle$ denotes the linear span.

For each $Q \in X_{m,d}$, let $T_Q X_{m,d} \subset \mathbb{P}^n$ denote the Zariski tangent space of $X_{m,d}$ at Q. The set $T_Q X_{m,d}$ is a projective space of dimension m. For any k distinct points $O_1, \dots, O_k \in \mathbb{P}^m$ set

$$W(O_1, \dots, O_k; d) := \langle \cup_{i=1}^{k} T_{\nu_d(O_i)} X_{m,d} \rangle \subseteq \mathbb{P}^n.$$

For each $r \in \mathbb{N}$ set $W(O_1, \dots, O_k; d)(r) := \{P \in W(O_1, \dots, O_k; d) : r_{X_{m,d}}(P) = r\}$ and $W(O_1, \dots, O_k; d)(\leq r) := \{P \in W(O_1, \dots, O_k; d) : r_{X_{m,d}}(P) \leq r\}$.

We prove the following result.

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Theorem 1.1 *Fix integers m, d, k such that $m \ge 2$, $d \ge 7$ and $2 \le k \le (d^2 - 10d + 17)/8$. Fix general $O_1, \dots, O_k \in \mathbb{P}^m$ and take any $r \in \{1, \dots, d-3\}$ and any $P \in W(O_1, \dots, O_k; d)(r)$. Then, there is a unique set $S \subseteq \{O_1, \dots, O_k\}$ such that $P \in \langle \nu_d(S) \rangle$ and $\sharp(S) = r$.*

In the set-up of Theorem 1.1, the set S is the only set evincing the rank of P (Proposition 2.4), i.e., $r_{X_{m,d}}(P) = r$ and S is the only set $A \subset \mathbb{P}^m$ with cardinality r such that $P \in \langle \nu_d(A) \rangle$. See Proposition 2.2 for a stronger statement if $m = 2$.

We recall that a finite set $S \subset \mathbb{P}^s$ is said to be *in linearly general position* if $\dim(\langle E \rangle) = \min\{s, \sharp(E) - 1\}$ for all $E \subseteq S$. When $m = 2$, Theorem 1.1 is quite good (Remark 2.3), but when $m \gg d$ it says almost nothing. For any $m \ge 2$, we prove the following result.

Theorem 1.2 *Fix integers m, d, k, r such that $m \ge 2$, $d \ge 7$, $1 \le r \le d - 3$ and $k < (m(d-2) + 4 - r)/2$. Fix a set $\{O_1, \dots, O_k\} \subset \mathbb{P}^m$ in linearly general position and take any $P \in W(O_1, \dots, O_k; d)(r)$. Then, there is a unique set $S \subseteq \{O_1, \dots, O_k\}$ such that $P \in \langle \nu_d(S) \rangle$ and $\sharp(S) = r$.*

Corollary 1.3 *Fix m, d, k, r and $(O_1, \dots, O_k) \in (\mathbb{P}^m)^k$ either as in Theorem 1.1 or as in Theorem 1.2. Let $M \subset W(O_1, \dots, O_k; d)$ be a general subspace of dimension $(m+1)k - 2 - r$. Then, $r_{X_{m,d}}(P) > r$ for all $P \in M$.*

We work over an algebraically closed base field $\mathbb{K}$ (see Remarks 2.5 and 3.5 for more general fields, Remark 3.6 for a discussion of the positive characteristic case).

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2 The set-up of Theorem 1.1

Remark 2.1 For every $O \in \mathbb{P}^n$ and every $Q \in T_{\nu_d(O)} X_{m,d}$, there is a degree two zero-dimensional scheme $A \subset \mathbb{P}^m$ such that $A_{\text{red}} = \{O\}$, $\deg(A) = 2$ and $Q \in \langle \nu_d(Z) \rangle$. Hence, for each $P \in W(O_1, \dots, O_k; d)$, there is a zero-dimensional scheme $Z \subset \mathbb{P}^m$ such that $Z_{\text{red}} = \{O_1, \dots, O_k\}$, each connected component of Z has degree two and $P \in \langle \nu_d(Z) \rangle$.

For each integer $t \ge 1$, the t-secant variety $\sigma_t(X_{m,d}) \subseteq \mathbb{P}^n$ of $X_{m,d}$ is the closure inside $\mathbb{P}^n$ of all linear spaces $\langle A \rangle$ with $A \subset X_{m,d}$ and $\sharp(A) = t$. The border rank $b_{X_{m,d}}(P)$ of $P \in \mathbb{P}^m$ is the first integer t such that $P \in \sigma_t(X_{m,d})$. When $b_{X_{m,d}}(P) \le d + 1$ there is a zero-dimensional scheme $Z \subset \mathbb{P}^m$ such that $\deg(Z) = b_{X_{m,d}}(P)$ and $P \in \langle \nu_d(Z) \rangle$ ([9], Proposition 11, [12], Lemma 2.16). We say that any such Z evinces the border rank of P.

We first do the case $m = 2$, because in this case [16] is a very powerful tool (which is also stated and proved in arbitrary characteristic).

Proposition 2.2 *Fix integers d, k such that $d \ge 7$ and $2 \le k \le (d^2 - 10d + 17)/8$. Fix general $O_1, \dots, O_k \in \mathbb{P}^2$. For each $i \ne j$, set $L_{i,j} := \langle \{O_i, O_j\} \rangle$.*

(a) *Fix $x \in \{1, \dots, d-3\}$ and $P \in W(O_1, \dots, O_k; d)(x)$. Then, there is $U \subseteq \{O_1, \dots, O_k\}$ such that $\sharp(U) = x$ and $P \in \langle \nu_d(U) \rangle$.*
(b) *Fix $P \in W(O_1, \dots, O_k; d)(d-2)$. Then, either there is $U \subseteq \{O_1, \dots, O_k\}$ such that $\sharp(U) = d-2$ and $P \in \langle \nu_d(U) \rangle$ or $P \in (\cup_{i<j} \langle \nu_d(L_{i,j}) \rangle)$.*
(c) *Fix $P \in \langle \nu_d(L_{i,j}) \rangle$ and assume $r_{X_{m,d}}(P) \ge d-2$ and $\mathrm{char}(\mathbb{K}) = 0$. Let v' (resp. v'') be the degree 2 zero-dimensional subscheme of $L_{i,j}$ with P_i (resp. P_j) as its support. Then, either $r_{X_{m,d}}(P) = d-2$, $br_{X_{m,d}}(P) = 4$ and $v' \cup v''$ evinces the border rank of P or $r_{X_{m,d}}(P) = d-1$, $br_{X_{m,d}}(P) = 3$ and either $v' \cup \{O_j\}$ or $v'' \cup \{O_i\}$ evince the border rank of P or $r_{X_{m,d}}(P) = d$, $br_{X_{m,d}}(P) = 2$ and either v' or v'' evince the border rank of P.*

Proof Fix $P \in W(O_1, \dots, O_k; d)(\le d-2)$ and set $r := r_{X_{2,d}}(P)$. Take any $A \subset \mathbb{P}^2$ evincing the rank of P, i.e., any finite set $A \subset \mathbb{P}^2$ such that $\sharp(A) = r_{2,d}(P)$ and $P \in \langle \nu_d(A) \rangle$. There is a zero-dimensional scheme $Z \subset \mathbb{P}^2$ such that $Z_{\text{red}} \subseteq \{O_1, \dots, O_k\}$, each connected component of Z has degree two and $P \in \langle \nu_d(Z) \rangle$ (Remark 2.1). Take $W \subseteq Z$ such that $P \in \langle \nu_d(W) \rangle$ and $P \notin \langle \nu_d(W') \rangle$ for each $W' \subsetneq W$. Set $w := \deg(W)$ and $w' := \sharp(W_{\text{red}})$. Since $W \subseteq Z$, we have $w \le 2w'$, each connected component of W has degree ≤ 2 and W_{red} is general in $\mathbb{P}^2$. In particular, W_{red} has general postulation, i.e., $h^0(\mathcal{I}_{W_{\text{red}}}(t)) = \max\{0, \binom{t+2}{2} - w'\}$ for all $t \in \mathbb{N}$. Take any line $L \subset \mathbb{P}^2$. We get $\deg(W \cap L) \le 4$ and that if $\deg(W \cap L) \ge 3$, then $\sharp(L \cap W_{\text{red}}) \ge 2$ (i.e., L is one of the lines $L_{i,j}$ and $W_{\text{red}} \supseteq \{O_i, O_j\}$).

(i) First assume $W = A$. Since A is reduced, W is reduced in this case. We have $A \subseteq \{O_1, \ldots, O_k\}$. We get the existence of $A \subseteq \{O_1, \ldots, O_k\}$ such that A evinces the rank of P. Take $U := A$ to prove parts (a) and (b) if $W = A$.

(ii) Now, assume $W \neq A$ and $r \leq d - 2$ [as in parts (a) and (b)]. To prove part (a), we need to find a contradiction if $r \neq d - 2$. To prove part (b), we need to prove that $P \in \langle \nu_d(L_{i,j}) \rangle$ for some i, j if $r = d - 2$ and $A \neq W$. We have $h^1(\mathcal{I}_{A \cup W}(d)) > 0$ ([6], Lemma 1). Let τ be the maximal integer t such that $h^1(\mathcal{I}_{A \cup W}(t)) > 0$. We just proved that $\tau \geq d$. Set $z := \deg(A \cup W)$ and $s := \lfloor \sqrt{z} \rfloor$. We have $z \leq d - 2 + 2k$ and $s \leq z/s$. Since $h^1(\mathcal{I}_{A \cup W}(d)) > 0$, we have $z \geq d + 2$ ([9], Lemma 34). Hence, $s \geq 3$.

Claim 1 $d \geq 2s + 3$.

Proof of Claim 1 Since, $s = \lfloor \sqrt{z} \rfloor$, to prove Claim 1 it is sufficient to prove the inequality $4z \leq d^2 - 6d + 9$. Since $z \leq 2k + d - 2$, it is sufficient to use the assumption $k \leq (d^2 - 10d + 17)/8$.

Since $z < (s+1)^2$, we have $z/s \leq s + 2$. Claim 1 gives $d \geq 2s - 1 \geq s - 3 + z/s$. Hence, $\tau \geq s - 3 + z/s$. Hence, we may apply [16], Corollaire 3, and get that either $\tau = s - 3 + z/s$ and $A \cup W$ is the complete intersection of a curve of degree s and a curve of degree z/s or there is an integer $t \in \{1, \ldots, s - 1\}$ and a curve $T \subset \mathbb{P}^2$ such that $\deg(T) = t$ and $\deg(T \cap (A \cup W)) \geq t(\tau - t + 3)$.

(ii.1) First assume $\tau = s - 3 + z/s$ and that $A \cup W$ is the complete intersection of a curve of degree s and a curve of degree z/s. In particular, W_{red} is contained in a curve of degree s. Since W_{red} has general postulation, we get $w' \leq (s^2 + 3s)/2$. Hence, $w \leq s^2 + 3s$. Since $r \leq d - 2$, we get $z \leq s^2 + 3s + d - 2$. We also have $d \leq \tau = s - 3 + z/s \leq s - 3 + s + 3 + (d - 2)/s$. Hence, $(s - 1)d \leq 2s^2 - 2$. Since $d \geq 2s + 3$ by Claim 1, we get a contradiction.

(ii.2) Now, assume the existence of an integer $t \in \{1, \ldots, s - 1\}$ and a curve $T \subset \mathbb{P}^2$ such that $\deg(T) = t$ and $\deg(T \cap (A \cup W)) \geq t(\tau - t + 3) \geq t(d - t + 3)$.

(ii.2.1) For each $x \in \mathbb{R}$ set $\psi_d(x) := 2x^2 - (x - 1)d - 2$. We have $\psi'_d(x) = 4x - d$. Hence, $\psi'_d(x) \leq 0$ if $x \leq d/4$ and $\psi'_d(x) \geq 0$ if $x \geq d/4$. We have $\psi_d(1) = 0$, $\psi_d(2) = 6 - d < 0$, $\psi(d/4) = d^2/8 - 3d^2/4 - 2 < 0$ and $\psi_d(s - 1) = 2(s - 1)^2 - (s - 2)d - 2 < 0$ by Claim 1. Hence, $\psi_d(x) < 0$ if $2 \leq x \leq s - 1$.

(ii.2.2) Since W_{red} has general postulation, we have $\sharp(W_{\mathrm{red}} \cap T) \leq (t^2 + 3t)/2$. Hence, $\deg(W \cap T) \leq t^2 + 3t$. Hence, $\deg((W \cup A) \cap T) \leq t^2 + 3t + d - 2$. Hence, $t^2 + 3t + d - 2 \geq t(d - t + 3)$, i.e., $\psi_d(t) \geq 0$. By step (ii.2.1) we have $t = 1$, i.e., there is a line $L \subset \mathbb{P}^2$ such that $\deg(L \cap (W \cup A)) \geq d + 2$. We saw before step (i) that $\deg(L \cap W) \leq 4$ and that if equality holds, then $L = L_{i,j}$ for some i, j. If $r \leq d - 3$, we get $\deg(W \cap L) \geq 5$, a contradiction, concluding the proof of part (a). If $r = d - 2$, we get $L = L_{i,j}$ for some i, j and $A \subset L_{i,j}$. Since $P \in \langle \nu_d(A) \rangle$, we get $P \in \langle \nu_d(L_{i,j}) \rangle$, concluding the proof of part (b).

(iii) Take the set-up of part (c). By concision (either [11], Sect. 3.1, or [13], Remark 2.3, or [23], Proposition 3.1, or [21], Exercise 3.2.2.2) $r_{X_{m,d}}(P)$ is the rank of P with respect to the rational normal curve $\nu_d(L_{i,j}) = X_{1,d}$. We also have $br_{X_{m,d}}(P) = br_{X_{1,d}}(P)$ ([11], Sect. 3.1). Hence, by the bivariate case ([9,15,22], Theorem 4.1) we know that $br_{X_{m,d}}(P) + r_{X_{m,d}}(P) = d + 2$ and that there is a zero-dimensional scheme $Z \subset L_{i,j}$ such that $P \in \langle \nu_d(Z) \rangle$, $P \notin \langle \nu_d(Z') \rangle$ for any $Z' \subsetneq Z$ and $\deg(Z) = br_{X_{1,d}}(P)$. Since $r_{X_{m,d}}(P) \geq d - 2$, we have $\deg(Z) \leq 4$. The proof of part (b) also gave $P \in \langle \nu_d(v' \cup v'') \rangle$. Let $w \subseteq v' \cup v''$ be a minimal subscheme such that $P \in \langle w \rangle$ of $v' \cup v''$. Since $d \geq 7 \geq \deg(Z) + \deg(w) - 1$, we have $h^1(\mathcal{I}_{Z \cup w}(d)) = 0$. Since $P \in \langle \nu_d(Z) \rangle \cap \langle w \rangle$, [6], Lemma 1, gives $Z \subseteq w$. Then, we write all subschemes γ of $v' \cup v'$ with degree 1, 2, 3, 4. Since $r \geq d - 2 > 2$ we exclude all cases in which γ is reduced, i.e., the cases $\gamma = \{O_i\}$, $\gamma = \{O_j\}$ and $\gamma = \{O_i, O_j\}$. □

Remark 2.3 Proposition 2.2 is quite strong, because if $d \geq 5$ and $k \geq \lceil (d + 2)(d + 1)/6 \rceil$, then $W(O_1, \ldots, O_k) = \mathbb{P}^n$ [1,2,10,14].

Proof of Theorem 1.1 The case $m = 2$ of Theorem 1.1 follows from the proof of Proposition 2.2. Hence, we may assume $m \geq 3$. Fix $r \in \{1, \ldots, d - 3\}$, any $P \in W(O_1, \ldots, O_k; d)(r)$ and any $A \subset \mathbb{P}^m$ evincing the rank of P. Remark 2.1 gives the existence of a minimal zero-dimensional scheme $W \subset \mathbb{P}^m$ such that $P \in \langle \nu_d(W) \rangle$, $W_{\mathrm{red}} \subseteq \{O_1, \ldots, O_k\}$ and each connected component of W has degree ≤ 2. Set $w := \deg(W)$. We need to prove that $A = W$. Assume that $W \neq A$. Since $P \notin \langle \nu_d(W') \rangle$ for any $W' \subsetneq W$ and any $W' \subsetneq A$, we have $h^1(\mathcal{I}_{A \cup W}(d)) > 0$ ([6], Lemma 1). Let $M \subset \mathbb{P}^m$ be a general subspace of dimension $m - 3$. Let $\ell : \mathbb{P}^m \setminus M \to \mathbb{P}^2$ be the linear projection from M. Since M is general, we have $M \cap (A \cup \{O_1, \ldots, O_k\}) = \emptyset$,

M intersects no line spanned by two of the points of $A \cup \{O_1, \dots, O_k\}$ or by a degree two subscheme of W. Hence, ℓ is defined in a neighborhood of $A \cup W$, $\ell|(A \cup W_{\mathrm{red}})$ is injective and ℓ send isomorphically onto its image each connected component of W. Hence, $\ell(A \cup W)$ is a scheme isomorphic to $A \cup W$ (as abstract schemes). Set $A' := \ell(A)$ and $W' := \ell(W)$. For general M, we may still assume that W'_{red} is formed by general points of $\mathbb{P}^2$.

Claim We have $h^1(\mathbb{P}^2, \mathcal{I}_{A' \cup W'}(d)) > 0$.

Proof of the Claim Assume $h^1(\mathbb{P}^2, \mathcal{I}_{A' \cup W'}(d)) = 0$. Since the linear projection from M induces an isomorphism between $A \cup W$ and $A' \cup W'$, we get that $A \cup W$ imposes $\deg(A \cup W)$ independent conditions to the linear subspace of $|\mathcal{O}_{\mathbb{P}^m}(d)|$ formed by the degree d cones with vertex containing M. Hence, $A \cup W$ imposes $\deg(A \cup W)$ independent conditions to $|\mathcal{O}_{\mathbb{P}^m}(d)|$, i.e., $h^1(\mathcal{I}_{A \cup W}(d)) = 0$, a contradiction.

By the Claim there is a minimal subscheme $W_1 \subseteq W'$ and a minimal subscheme $A_1 \subseteq A'$ such that $h^1(\mathbb{P}^2, \mathcal{I}_{A_1 \cup W_1}(d)) > 0$. We have $\sharp((W_1)_{\mathrm{red}}) \le k \le (d^2 - 10d + 17)/8$ and $\sharp(A_1) \le \sharp(A') = \sharp(A) \le d - 3$. Moreover, $(W_1)_{\mathrm{red}}$ is general in $\mathbb{P}^2$. We are in the set-up of part (a) of Proposition 2.2 and we adapt step (ii) of its proof. Let τ be the maximal integer t such that $h^1(\mathbb{P}^2, \mathcal{I}_{A_1 \cup W_1}(t)) > 0$. Set $z := \deg(A_1 \cup W_1)$ and $s := \lfloor \sqrt{z} \rfloor$. We Claim 1 and parts (ii.2.1) and (ii.2.2) of the proof of Proposition 2.2 give the existence of a line $L \subset \mathbb{P}^2$ such that $\deg(L \cap (A_1 \cup W_1)) \ge d + 2$. Since $(W_1)_{\mathrm{red}}$ is general and each connected component of W_1 has degree ≤ 2, we get $\deg(L \cap W) \le 4$. Hence, $\deg(A_1) \ge \deg(A_1 \cap L) \ge d - 2$, a contradiction. □

Proposition 2.4 *Fix P and S as in the statement of Theorem 1.1. Then, S is the only subset of $\mathbb{P}^m$ evincing the rank of P.*

Proof Since $P \in \langle \nu_d(S) \rangle$, $\sharp(S) = r$ and $r_{X_{m,d}}(P) = r$, then S is one of the sets evincing the rank of P. Assume the existence of a set $A \ne S$ such that A evinces the rank of P. By the definition of rank, we have $P \notin \langle \nu_d(E) \rangle$, if E is either a proper subset of A or a proper subset of S. Hence, $h^1(\mathcal{I}_{A \cup S}(d)) > 0$ ([6], Lemma 1). Since $\deg(A \cup S) \le 2r \le 2d + 1$, there is a line $L \subset \mathbb{P}^m$ such that $\sharp(L \cap (A \cup S)) \ge d + 2$ ([9], Lemma 34). Since $O_1, \dots, O_k$ are general in $\mathbb{P}^m$, we have $\sharp(S \cap L) \le 2$. Hence, $\sharp(L \cap A) > \sharp(S \cap L)$. The proof of [6], Theorem 2.2 (alternatively, apply [7], Lemma 5.1), gives $A \setminus A \cap L = S \setminus S \cap L$. Hence, $\sharp(A) > \sharp(S)$, a contradiction. □

Remark 2.5 Fix any infinite field K and call $\overline{K}$ its algebraic closure. We claim that Proposition 2.2 [with the assumption that either $\mathrm{char}(K) = 0$ or $\mathrm{char}(K) > d$ for part (c)] and Theorem 2.2 are true for sufficiently generic $O_1, \dots, O_k \in \mathbb{P}^m(K)$, i.e., for all k-ples $(O_1, \dots, O_k)$ in a subset of $\mathbb{P}^m(K)^k$ which is Zariski dense in $\mathbb{P}^m(\overline{K})^k$. Since K is infinite, $\mathbb{P}^m(K)$ is Zariski dense in $\mathbb{P}^m(\overline{K})$. Hence, by [1,2,10,14] (as in Remark 2.3) there is a $\Delta \subseteq \mathbb{P}^m(K)^k$ such that Δ is Zariski dense in $\mathbb{P}^m(\overline{K})^k$ and $\dim(W(O_1, \dots, O_k; d)) = \min\{k(m + 1) - 1, \binom{m+d}{m} - 1\}$ for all $(O_1, \dots, O_k) \in \Delta$. The assumptions on k give $k(m+1) \le \binom{m+d}{m}$. Fix an integer $d \ge 7$. Let Λ be the set of all $(Q_1, \dots, Q_k) \in \mathbb{P}^2(\overline{K})^k$ such that $h^0(\mathbb{P}^2, \mathcal{I}_{\{Q_1,\dots,Q_k\}}(t)) = \max\{0, \binom{t+2}{2} - k\}$ for all $t \in \{1, \dots, d\}$. Since Λ is a non-empty open subset of $\mathbb{P}^2(\overline{K})^k$, $\Delta \cap \Lambda$ is non-empty and Zariski dense in $\mathbb{P}^m(\overline{K})$. The proof of Proposition 2.2 works for all $(O_1, \dots, O_k) \in \Delta \cap \Lambda$. Now, assume $m > 2$. Let $M \subset \mathbb{P}^m$ be any $(m-3)$-dimensional linear subspace defined over K and call ℓ the linear projection from M. Since K is infinite, we may find M disjoint from the finitely many lines spanned by a degree two subscheme of $A \cup W$. Since M is defined over K, we have $\ell(\mathbb{P}^m(K) \setminus M \cap \mathbb{P}^m(K)) = \mathbb{P}^2(K)$. The set Γ of all $(O_1, \dots, O_k) \in \mathbb{P}^m(K)^k$ such that $O_i \notin M$ for all i and $(\ell(O_1), \dots, \ell(O_k)) \in \Delta \cap \Lambda$ is Zariski dense in $\mathbb{P}^m(\overline{K})$. The proof of Theorem 1.1 works for every $(O_1, \dots, O_k) \in \Gamma$.

3 Proofs of Theorem 1.2 and Corollary 1.3

For any zero-dimensional scheme $Z \subset \mathbb{P}^m$ and any hyperplane $H \subset \mathbb{P}^m$, the residual scheme $\mathrm{Res}_H(Z)$ of Z with respect to H is the closed subscheme of $\mathbb{P}^m$ with $\mathcal{I}_Z : \mathcal{I}_H$ as its ideal sheaf. We have $\mathrm{Res}_H(Z) \subseteq Z$ and $\deg(Z) = \deg(\mathrm{Res}_H(Z)) + \deg(H \cap Z)$.

We need the following obvious lemma whose proof is omitted.

Lemma 3.1 *Let $B \subset \mathbb{P}^x$, $x \ge 1$, be a linearly independent set. Take any zero-dimensional scheme $A \subset \mathbb{P}^x$ such that $A_{\mathrm{red}} = B$ and $\deg(A) \le \sharp(B) + 1$. Then, $h^1(\mathcal{I}_A(2)) = 0$.*

Lemma 3.2 *Fix an integer $t \ge 2$, a finite set $S \subset \mathbb{P}^s$ which is linearly independent and spanning $\mathbb{P}^s$, a zero-dimensional scheme $W \supset S$ with $\deg(W) = s + 2$ and a set $A \subset \mathbb{P}^s$ such that $\sharp(A) \le t - 2$. Then, $h^1(\mathcal{I}_{A \cup W}(t)) = 0$.*

Proof If $s = 1$, then the lemma is true, because $\deg(A \cup W) \le t + 1$ in this case. Hence, we may assume $s \ge 2$ and use induction on s. Take a hyperplane $H \subset \mathbb{P}^s$ spanned by s of the points of S, with the only restriction that if W is not reduced then H contains the support of the only unreduced connected component of W. The inductive assumption gives $h^1(H, \mathcal{I}_{(A\cup W)\cap H,H}(t)) = 0$. Look at the residual exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_H(A\cup W)}(t-1) \to \mathcal{I}_{A\cup W}(t) \to \mathcal{I}_{H\cap(A\cup W),H}(t) \to 0 \tag{1}$$

The scheme $\mathrm{Res}_H(W)$ has degree at most 2 and it is reduced. Since $\mathrm{Res}_H(A \cup W)$ is the union of $\mathrm{Res}_H(W)$ and $A \setminus A \cap H$, we have $\deg(\mathrm{Res}_H(A \cup W)) \le t$. Hence, $h^1(\mathcal{I}_{\mathrm{Res}_H(A\cup W)}(t-1)) = 0$ ([9], Lemma 34). Hence, (1) gives $h^1(\mathcal{I}_{A\cup W}(t)) = 0$. □

The following two elementary lemmas are very classical and in characteristic zero stronger results are known (e.g. [8], Lemma 1.8). However, the statements and proofs must be characteristic free to hope any application to codes over a finite field.

Lemma 3.3 *Let $Z \subset \mathbb{P}^s$, $s \ge 1$, be a zero-dimensional scheme such that $S := Z_{\mathrm{red}}$ is linearly independent in $\mathbb{P}^s$ and each connected component of Z has degree ≤ 2. Then, $h^1(\mathcal{I}_Z(3)) = 0$.*

Proof We have $\sharp(S) \le s + 1$. The lemma is true if $s = 1$, because $\deg(Z) \le 4$ if $s = 1$. Hence, we may assume $s \ge 2$ and that the lemma is true for lower dimensional projective spaces. Let $H \subset \mathbb{P}^s$ be a hyperplane such that $\sharp(S \cap H)$ is maximal, i.e., take any $H \supset S$ if $\sharp(S) \le s$ and any H spanned by s of the point of S if $\sharp(S) = s + 1$. Look at the residual exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_H(Z)}(2) \to \mathcal{I}_Z(3) \to \mathcal{I}_{Z\cap H,H}(3) \to 0 \tag{2}$$

The inductive assumption gives $h^1(H, \mathcal{I}_{Z\cap H,H}(3)) = 0$. First assume $\sharp(S) \le s$. In this case, $\mathrm{Res}_H(Z)$ is a reduced scheme contained in S. Hence, $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(2)) = 0$. Now, assume $\sharp(S) = s+1$. In this case, $\mathrm{Res}_H(Z)$ is a scheme whose reduction, B, is contained in S and with at most one unreduced connected component (the component of Z not intersecting H). Lemma 3.1 gives $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(2)) = 0$. Hence, (2) gives $h^1(\mathcal{I}_Z(3)) = 0$. □

Lemma 3.4 *Fix an integer $d \ge 3$, a finite set $A \subset \mathbb{P}^s$, $s \ge 1$, and a zero-dimensional scheme $W \subset \mathbb{P}^s$ such that $\sharp(A) \le d - 3$, W_{red} is linearly independent, and each connected component of W has degree ≤ 2. Then, $h^1(\mathcal{I}_{A\cup W}(d)) = 0$.*

Proof The lemma is true if $s = 1$, because $\deg(W \cup A) \le 4 + d - 3$ if $s = 1$. Hence, we may assume $s \ge 2$ and use induction on s. The lemma is true if $A = \emptyset$ (Lemma 3.3). Hence, we may assume $A \ne \emptyset$ and in particular $d \ge 4$. Taking a scheme $W_1 \supseteq W$, we reduce to the case $\sharp(W_{\mathrm{red}}) = s + 1$. Assume $h^1(\mathcal{I}_{A\cup W}(d)) > 0$. Let $H \subset \mathbb{P}^s$ be any hyperplane containing s points of W_{red}. Since $W_{\mathrm{red}} \cap H$ is linearly independent, the inductive assumption gives $h^1(H, \mathcal{I}_{H\cap(A\cup W),H}(d)) = 0$. Hence, (1) shows that it is sufficient to prove $h^1(\mathcal{I}_{\mathrm{Res}_H(A\cup W)}(d-1)) = 0$. The scheme $\mathrm{Res}_H(A \cup W)$ is the union of $W' := \mathrm{Res}_H(W)$ and the set $A' := A \setminus A \cap H$. The scheme W' has as its reduction a subset of W_{red} and at most one of its connected components is not reduced (in this case it has degree two if it exists). Hence, there is a hyperplane $M \subset \mathbb{P}^s$ such that $\deg(\mathrm{Res}_M(W')) \le 1$. We have $\sharp(A' \cap M) \le \sharp(A) \le d - 3$. Lemma 3.2 gives $h^1(M, \mathcal{I}_{M\cap(A'\cup W')}(d-1)) = 0$. The set $\mathrm{Res}_M(A' \cup W')$ has cardinality at most $d - 2$. Hence, $h^1(\mathcal{I}_{\mathrm{Res}_M(A'\cup W')}(d-2)) = 0$. A residual sequence like (1) for $t = d - 1$ gives $h^1(\mathcal{I}_{\mathrm{Res}_H(A\cup W)}(d-1)) = 0$. □

Proof of Theorem 1.2 Fix any $A \subset \mathbb{P}^m$ evincing the rank of P. Remark 2.1 gives the existence of a minimal zero-dimensional scheme $W \subset \mathbb{P}^m$ such that $P \in \langle \nu_d(W)\rangle$, $W_{\mathrm{red}} \subseteq \{O_1, \dots, O_k\}$ and each connected component of W has degree ≤ 2. Set $w := \deg(W)$. We need to prove that $A = W$. Assume that $W \ne A$. Since $P \notin \langle \nu_d(W')\rangle$ for any $W' \subsetneq W$ and any $W' \subsetneq A$, we have $h^1(\mathcal{I}_{A\cup W}(d)) > 0$ ([6], Lemma 1).

Let $\mathcal{S}_0$ be the set of all hyperplanes $M \subset \mathbb{P}^m$ containing at least one point of A. Set $B_0 := A \cup W$, $A_0 := A$ and $W_0 := W$. Fix $H_1 \in \mathcal{S}_0$ such that $\deg(H_1 \cap (A \cup W))$ is maximal among all $H_1 \in \mathcal{S}_0$. Set $B_1 := \mathrm{Res}_{H_1}(B_0)$, $W_1 := \mathrm{Res}_{H_1}(W_0)$ and $A_1 := \mathrm{Res}_{H_1}(A_0) = A \setminus A \cap H_1$. If $A_1 = \emptyset$, then let $\mathcal{S}_1$ be the set of all hyperplanes of $\mathbb{P}^m$. If $A_1 \ne \emptyset$, then let $\mathcal{S}_1$ be the set of all hyperplanes of $\mathbb{P}^m$ containing at least one point of A_1. For all integers $i \ge 2$ defined recursively the hyperplanes $H_i \subset \mathbb{P}^m$, the integer a_i, the zero-dimensional schemes B_i, W_i, A_i and the set $\mathcal{S}_i$ of hyperplanes of $\mathbb{P}^m$ in the following way. Let $H_i \in \mathcal{S}_{i-1}$ be such that $a_i := \deg(H_i \cap B_{i-1})$ is maximal among all hyperplanes of $\mathcal{S}_{i-1}$ and set $B_i := \mathrm{Res}_{H_i}(B_{i-1})$,

$W_i := \mathrm{Res}_{H_i}(W_{i-1})$ and $A_i := \mathrm{Res}_{H_i}(A_{i-1})$. Hence, $B_i = A_i \cup W_i$. If $A_i = \emptyset$, then let $\mathcal{S}_i$ be the set of all hyperplanes of $\mathbb{P}^m$. If $A_i \neq \emptyset$, then let $\mathcal{S}_i$ be the set of all hyperplanes of $\mathbb{P}^m$ containing at least one point of A_i. Every zero-dimensional scheme $E \subset \mathbb{P}^m$ of degree $\leq m$ is contained in a hyperplane. Hence, if $a_i \leq m-1$, then $B_{i-1} \subset H_i$ and $a_{i+1} = 0$. Notice that if $A_{i-1} \neq \emptyset$, then $A_i \subsetneq A_{i-1}$. Since $\sharp(A_0) = r \leq d-3$, we get that for each $i > 0$ either $A_i = \emptyset$ or $\sharp(A_i) \leq d-3-i$. For all $i \geq 0$, we have the following residual exact sequences

$$0 \to \mathcal{I}_{B_{i+1}}(d-i-1) \to \mathcal{I}_{B_i}(d-i) \to \mathcal{I}_{B_i \cap H_{i+1}, H_{i+1}}(d-i) \to 0 \tag{3}$$

Since $h^1(\mathcal{I}_{B_0}(d)) > 0$, the exact sequences (3) for $i \geq 0$ give the existence of a minimal integer $e \geq 0$ such that $h^1(H_{e+1}, \mathcal{I}_{B_e \cap H_{e+1}, H_{e+1}}(d-e)) > 0$. Since $(W_i)_{\mathrm{red}} \cap H_{i+1}$ is linearly independent in H_{i+1} and either $A_i = \emptyset$ or $\sharp(A_i) \leq d-3-i$, Lemma 3.4 gives $e \geq d-2$. Assume for the moment $e \geq d-1$. We get $2k + r \geq w + r \geq m(d-1)$, a contradiction. Now, assume $e = d-2$. Since $h^1(H_{d-1}, \mathcal{I}_{B_{d-2} \cap H_{d-1}}(2)) > 0$, we have $\deg(B_{d-2} \cap H_{d-1}) \geq 4$ ([9], Lemma 34). Hence, $2k + r \geq w + r \geq m(d-2) + 4$, contradicting the assumption $k < (m(d-2) + 4 - r)/2$. □

Proof of Corollary 1.3 Let $M \subset W(O_1, \ldots, O_k)$ be a general linear subspace with codimension at least $r+1$. Since M is general, we have $M \cap \langle \nu_d(S) \rangle = \emptyset$ for all $S \subseteq \{O_1, \ldots, O_k\}$ with $\sharp(S) = \min\{k, r\}$. Hence, to conclude the proof of Corollary 1.3 it is sufficient to prove that $\dim(W(O_1, \ldots, O_k)) = (m+1)k - 1$. This is true in the set-up of Theorem 1.1 by a weak form of a theorem of Alexander and Hirschowitz ([1,2,10,14]). Now, we take the set-up of Theorem 1.2. By [14], Lemma 4, it is sufficient to prove $h^1(\mathbb{P}^m, \mathcal{I}_Z(d)) = 0$ for each zero-dimensional scheme $Z \subset \mathbb{P}^m$ such that $Z_{\mathrm{red}} \subseteq \{O_1, \ldots, O_k\}$ and each connected component of Z as degree at most 2. Repeat the proof of Theorem 1.2 taking $A = \emptyset$. □

Remark 3.5 Theorem 1.2 is true over any field K for which there are k distinct points $O_1, \ldots, O_k \in \mathbb{P}^m(K)$ in linearly general position, just because we defined the rank using subsets of $X_{m,d}(\overline{K})$. The existence of k points of $\mathbb{P}^m(K)$ in linearly general position is obvious (for arbitrary k) if K is infinite. If K is finite, then it is sufficient to assume $k \leq \sharp(K) + 1$ [[19], Theorem 27.5.1 (iv)]; if $m = 2$ and $\sharp(S)$ is even we may even allow the case $k = \sharp(K) + 2$ ([19], Eq. (27.2). Hence, the statement of Theorem 1.2 is true for an arbitrary field K, with the only restriction that $m \geq 3$ if K is finite (but it may be an empty statement when K is finite if $k \geq \sharp(K) + 2$).

Remark 3.6 Assume $\mathbb{K}$ algebraically closed with characteristic $p > 0$. We fix the degree d of the homogeneous polynomial we are interested in and hence we fix ν_d. Fix any integer $m > 0$. We take as the definition of rank of $P \in \mathbb{P}^n$, $n := \binom{m+d}{m} - 1$, the $X_{m,d}$-rank, i.e., the rank with respect to the Veronese variety $X_{m,d} = \nu_d(\mathbb{P}^m)$. If $p > d$ (but only if $p > d$), we may translate this definition for a homogenous polynomial $f \neq 0$ as the minimal number of summands of d-powers of linear forms needed to obtain f. With our definition in terms of $X_{m,d}$-rank if $p > d$, then the case $m = 1$ is true ([20], Theorem 1.44), but the case $p \leq d$ fails (but it fails in a controlled way ([3]); for instance if $p = d = 2$, there is a unique point of $\mathbb{P}^2$ (the strange point of the smooth conic $X_{1,2} \subset \mathbb{P}^3$) with $X_{1,2}$-rank 3). In the set-up of Proposition 2.2 and in many other places, one can substitute the characteristic zero quotations of concisions with an explicit proof of that particular case using [11], Lemma 1, and then play as in the proof of Theorem 1.2; in the plane this game should be substituted with [16], as we did in step (ii) of the proof of Proposition 2.2. If $p > d$, part (c) of Proposition 2.2 is true. If $p \leq d$ instead of part (c) of Proposition 2.2, one can give the following statement whose proof follows from [7], Lemma 5.1, or [16].

As in part (c) of Proposition 2.2 take $d \geq 7$ and $P \in \langle \nu_d(L_{i,j}) \rangle$ with $r_{X_{2,d}}(P) \geq d-2$. Then, $b_{X_{2,d}}(P) = b_{X_{1,d}}(P)$, $r_{X_{2,d}}(P) = r_{X_{1,d}}(P)$, every subscheme of $\mathbb{P}^2$ evincing the border rank of P with respect to $X_{2,d}$ is contained in $L_{i,j}$ and every subset of $\mathbb{P}^2$ evincing the rank of P with respect to $X_{2,d}$ is contained in $L_{i,j}$.

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